CERTAIN PARTICULAR SOLUTIONS OF EQUATIONS FOR THE AXISYMMETRICAL PROBLEM IN THE THEORY OF IDEAL PLASTICITY AND GENERALIZATION OF PRANDTL' S SOLUTION OF COMPRESSION OF PLASTIC LAYER BETWEEN TWO ROUGH PLATES

(NEKOTORYE CHASTNYE RESHENIIA URAVNENII OSESIMMETRICHNOI ZADACHI TEORII IDEAL'NOI PLASTICHNOSTI I OBOBSHCHENIE RESHENIIA PRANDTLIA O SZHATII PLASTICHESKOGO SLOIA DVUMIA SHEROKHOVATYMI PLITAMI)

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In the theory of ideal plasticity the finding of closed particular solutions is undoubtedly of interest. Several such solutions for a case of plane problem have been pointed out and investigated by: L. Prandtl; A. Nadai; H. Hencky; C. Caratheodory and E. Schmidt; S.L. Sobolev; S.G. Mikhlin; V.V. Sokolovskii; R. Hill and others. These solutions are available, for example in monographs [1-3].

This paper considers certain particular solutions of the axisymmetrical problem in the theory of ideal plasticity under Mises and Tresca-St. Venant plasticity conditions and the associated laws of plastic flow.

Note that detailed investigations of particular solutions of the axisymmetrical problem which describe plastic stressed state in a converging channel are credited to Sokolovskii [1] and to Shield [4,5].

1. Following the Mises conditions of plasticity, the equations for the axisymmetrical problem in cylindrical coordinates appear as

$$\frac{\partial \sigma_\rho}{\partial \rho} + \frac{\partial \tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_z}{\rho} = 0, \qquad \frac{\partial \tau_{\rho z}}{\partial \rho} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} = 0 \tag{1.1}$$

$$(\sigma_\rho - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_\rho)^2 + 6\tau_{\rho z} = 6 \tag{1.2}$$

$$\varepsilon_\rho = \frac{\partial u}{\partial \rho} = \lambda(2\sigma_\rho - \sigma_\theta - \sigma_z) \qquad \varepsilon_\theta = \frac{u}{\rho} = \lambda(2\sigma_\theta - \sigma_z - \sigma_\rho)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = \lambda(2\sigma_z - \sigma_\rho - \sigma_\theta), \qquad 2\gamma_{\rho z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial \rho} = 6\tau_{\rho z} \tag{1.3}$$

where all the quantities are assumed to be dimensionless. Components of the stress are referred to a constant which is on the right-hand side of the plasticity condition; coordinates are referred to some characteristic length; and the displacement velocities are referred to some characteristic velocity.

I. The simplest particular solution may be indicated for $\tau_{\rho z} = w = 0$. This corresponds to a very well-explored axisymmetrical state of stress for plane strain.

II. As shown by Hill [3] the components of stress and displacement velocity, which satisfy relationships (1.1) to (1.3), may be given in the form:

$$\tau_{\rho z} = \rho, \quad \sigma_\rho = \sigma_\theta = -2z + C_1, \quad \sigma_z = -2z - \mu_1 \sqrt{3(1-\rho^2)} + C_1 \quad (1.4)$$

$$u = -\rho, \qquad w = 2z + 2\sqrt{3}\mu_1 \sqrt{1-\rho^2} + C_2$$

where C_1, C_2 are constants and $\mu_1 = \operatorname{sign}(\sigma_\rho - \sigma_z)$.

Hill applied this solution for the investigation of the squeezing of plastic material from contractable cylindrical housing with rough surface. We observed that this solution was analogous to a cycloidal solution introduced by Prandtl for a mass compressed between rough plates.

III. One may give a solution, also analogous to Prandtl's cycloidal solution, which corresponds to compression of a plastic material into a diverging rough cylindrical tube. In fact, put

$$\tau_{\rho z} = \frac{1}{\rho}, \qquad u = \frac{1}{\rho} \quad (1.5)$$

Then from relationships (1.3) it follows that

$$\sigma_z = \frac{1}{2}(\sigma_\rho + \sigma_\theta)$$

and the plasticity condition (1.2) will assume the form:

$$(\sigma_\rho - \sigma_\theta)^2 + 4\tau_{\rho z}^2 = 4 \quad (1.6)$$

From equations (1.6) and (1.5) we obtain

$$\sigma_\rho - \sigma_\theta = 2\mu_2 \sqrt{1 - \frac{1}{\rho^2}}, \qquad \mu_2 = \operatorname{sign}(\sigma_\rho - \sigma_\theta) \quad (1.7)$$

Substituting expression (1.7) into the first of equilibrium equations (1.1), we find

$$\sigma_\rho = 2\mu_2\left[\sqrt{1 - \frac{1}{\rho^2}} - \ln\left(\rho + \sqrt{\rho^2 - 1}\right)\right] + f(z) \quad (1.8)$$

From this it follows that

$$\sigma_\theta = -2\mu_2 \ln(\rho + \sqrt{\rho^2 - 1}) + f(z) \tag{1.9}$$

From the second of the equilibrium equations (1.1), and from (1.5), we find that

$$\sigma_z = \varphi(\rho)$$

Therefore, one should put $f(z) = C_1$ in relationships (1.8) and (1.9).

Obviously

$$\sigma_z = \mu_2 \left[\sqrt{1 - \frac{1}{\rho^2}} - 2\ln(\rho + \sqrt{\rho^2 - 1}) \right] + C_1$$

In order to find the displacement velocity w, we take the second and the fourth of the equations (1.3). On integration we obtain

$$w = -2\mu_2 \cos^{-1} \frac{1}{\rho} + C_2$$

IV. The most general analog of Prandtl's cycloidal solution of an axisymmetrical problem appears to be a solution which contains as particular cases the solutions presented in I and III. Let

$$\tau_{\rho z} = m_1\rho + \frac{m_2}{\rho}, \qquad u = n_1\rho + \frac{n_2}{\rho} \tag{1.10}$$

where m_1, m_2, n_1, n_2 are constants.

The first two equations (1.3) will be written in the form

$$n_1 - \frac{n_2}{\rho^2} = \lambda(2\sigma_\rho - \sigma_\theta - \sigma_z), \qquad n_1 + \frac{n_2}{\rho^2} = \lambda(2\sigma_\theta - \sigma_z - \sigma_\rho)$$

from which it is easy to obtain

$$\sigma_z = \frac{3n_1\rho^2}{2n_2}(\sigma_\rho - \sigma_\theta) + \frac{1}{2}(\sigma_\rho + \sigma_\theta) \tag{1.11}$$

Substituting expression (1.11) into plasticity condition (1.2), we obtain

$$\sigma_\rho - \sigma_\theta = \frac{2\mu_2 n_2}{\rho} \sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}} \tag{1.12}$$

From (1.10), (1.12), and from the first equation (1.1), determine

$$\sigma_\rho = -2\mu_2 n_2 \int \sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}} \frac{d\rho}{\rho^2} + f_1(z) \tag{1.13}$$

$$\sigma_\theta = -2\mu_2 n_1 \int \sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}} \frac{d\rho}{\rho^2} - \frac{2\mu_2 n_2}{\rho} \sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}} + f_1(z)$$

From (1.10), and from the second equation of (1.1), we find that

$$\sigma_z = -2m_1 z + \varphi_1(\rho) \tag{1.14}$$

Employing equations (1.11)-(1.14), we find that

$$\sigma_z = -2m_1 z + \mu_2\left(3n_1\rho - \frac{n_2}{\rho}\right)\sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}} -$$
$$-2\mu_2 n_2 \int \sqrt{\frac{\rho^2 - (m_1\rho^2 + m_2)^2}{n_2^2 + 3n_1^2\rho^4}}\frac{d\rho}{\rho^2} + C_1$$
$$f_1(z) = -2m_1 z + C_1$$

Having taken a ratio of the second and the fourth equations (1.3), we obtain

$$w = 6\int \frac{\tau_{\rho z} u\, d\rho}{\rho\,(2\sigma_\theta - \sigma_z - \sigma_\rho)} + f_2(z) \tag{1.15}$$

On the other hand, from the incompressibility equation it follows that

$$w = -2n_1 z + \varphi_2(\rho) \tag{1.16}$$

By comparing expressions (1.15) and (1.16) we find that

$$f_2(z) = 2n_1 z + C_2$$

The solution obtained corresponds to the pressing of a plastic cylindrical layer by rough coaxial cylindrical surfaces. Such a process of punching appears to be hypothetical, although if one investigates cylinders of sufficiently large radius, then the plastic material will be found in approximately the condition of being pressed by two parallel cylindrical surfaces.

In further discussions it is convenient to go over to a system of coordinates x^*, y^*, z^*; assuming the z^* axis to be perpendicular to the x^*y^* plane.

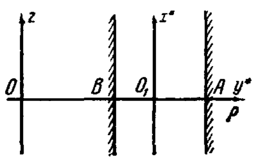

Fig. 1.

From the figure we have

$$OO_1 = R, \quad AO_1 = h_1, \quad BO_1 = h_2, \quad AB = 2h, \quad x^* = z, \quad y^* = \rho - R$$

We select the quantity h to represent the characteristic linear dimension, preserve the notations for dimensionless quantities R, h_1, h_2 and omit the stars for coordinates x^*, y^*, z^*.

Let us suppose that on the surfaces the shear stress τ_{xy} assumes the maximum values; then

$$m_1(R+h_1)+\frac{m_2}{R+h_1}=-1, \qquad m_1(R-h_2)+\frac{m_2}{R-h^2}=1 \quad (1.17)$$

The value of R will be determined from the condition

$$m_1 R+\frac{m_2}{R}=0 \tag{1.18}$$

From equations (1.17) and (1.18) we obtain

$$m_1=-\frac{1}{2}, \qquad m_2=\frac{R^2}{2} \qquad R=\frac{h_1 h_2}{h_1-h_2} \tag{1.19}$$

Assuming that the radius of the outer surface is decreasing with unit velocity, and the radius of the interior surface is increasing with the same velocity, we obtain analogously

$$n_1=-\frac{1}{2}, \qquad n_2=\frac{1}{2}R^2 \tag{1.20}$$

Let us consider a case of sufficiently large radius R. Having assumed $\delta = 1/R$, we neglect all the quantities which contain terms of δ^2 and higher.

After denoting $h_1 = 1 + \delta_1$, $h_2 = 1 - \delta_1$, from (1.19) we find that $\delta_1 = 1/2\ \delta h^2$.

Simplifying relationships (1.11)-(1.15), we obtain

$$\tau_{xy}=-y-\delta\frac{y^2}{2}$$

$$\sigma_x=x-2\mu_2\sqrt{1-y^2}+\frac{\mu_2\delta}{2}\left[\sin^{-1}y+\frac{y(1-3y^2)}{\sqrt{1-y^2}}\right]+C_1$$

$$\sigma_y=x+\frac{\mu^2\delta}{2}(y\sqrt{1-y^2}+\sin^{-1}y)+C_1$$

$$\sigma_z=x-\mu_2\sqrt{1-y^2}+\frac{\mu^2\delta}{2}\left[\sin^{-1}y+\frac{y(4-5y^2)}{\sqrt{1-y^2}}\right]+C_1$$

$$u=-y+\delta\frac{y^2}{2}, \qquad w=x+\mu_2\delta\left[3\sin^{-1}y-\frac{y(3-2y^2)}{\sqrt{1-y^2}}\right]+C_2$$

Relationships as given by Prandtl are obtained when $\delta = 0$, $\mu = -1$. Equations of slip lines are easily obtained by identifying them with the lines of action of maximum shear stresses.

2. In employing Tresca-St. Venant's conditions of plasticity, we will confine ourselves to the consideration of a case when the plastic state of stress corresponds to an edge of the prism which interprets the Tresca-St. Venant condition of plasticity in the space of principal stresses.

In other words, we will presuppose that the conditions of complete plasticity are satisfied.

The validity of this assumption is based on the development of complete solutions of problems.

The condition of plasticity will have the form

$$\frac{1}{4}(\sigma_\rho - \sigma_z)^2 + \tau_{\rho z}^2 = 1, \qquad \sigma_\theta = \frac{1}{2}(\sigma_\rho + \sigma_z) + 1 \tag{2.1}$$

For determining the field of velocities there will be equations

$$\dot\varepsilon_\rho + \dot\varepsilon_\theta + \dot\varepsilon_z = 0, \qquad \frac{\varepsilon_\rho - \varepsilon_z}{\sigma_\rho - \sigma_z} = \frac{\gamma_{\rho z}}{\tau_{\rho z}} \tag{2.2}$$

Equations determining the plastic state of stress under conditions of complete plasticity are statically determinate and belong to the hyperbolic type. It is known that if a transformation of variables is performed

$$\sigma_\rho = \omega + \sin 2\psi, \qquad \sigma_z = \omega - \sin 2\psi, \qquad \tau_{\rho z} = \cos 2\psi$$

then the equations of characteristics of initial equations (1.1), (2.1) assume the form

$$d\rho - \tan\psi\, dz = 0, \qquad d\rho + \cot\psi\, dz = 0 \tag{2.3}$$

whereby the characteristics coincide with the slip lines (the lines of maximum shear).

Since

$$\tan\psi = \sqrt{\frac{1 - \cos 2\psi}{1 + \cos 2\psi}}$$

then equations (2.3) may be rewritten in the form

$$\frac{d\rho}{dz} = \sqrt{\frac{1 - \tau_{\rho z}}{1 + \tau_{\rho z}}}, \qquad \frac{d\rho}{dz} = -\sqrt{\frac{1 + \tau_{\rho z}}{1 - \tau_{\rho z}}} \tag{2.4}$$

I. For such a case the simplest solution is credited to Hencky [6], who originated an application of the condition of complete plasticity in the theory of ideal plasticity:

$$\tau_{\rho z} = 0, \qquad \sigma_\rho = (1 - \mu_1)\ln\rho + C_1$$

$$\sigma_z = (1 - \mu_1)\ln\rho - 2\mu_1 + C_1 \tag{2.5}$$

Equations (2.2) for finding velocities of displacements assume the form

$$\frac{z\varrho}{m\varrho} + \frac{\partial}{n} + \frac{\partial\varrho}{n\varrho} = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial \rho} = 0$$

In such a case the characteristics appear as straight lines

$$\rho + z = \text{const}$$

It will be shown that solution (2.5) may not be applied to an investigation of the state of stress in thick-walled tubes for plane deformation. Consider an increment of work by stresses done in an increment of plastic deformation:

$$dA = \sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2 + \sigma_3 d\varepsilon_3$$

If the condition of complete plasticity $\sigma_1 = \sigma_2 = \sigma_3 \pm 2$ exists, then $dA = \pm\, 2\, d\epsilon_3$, and consequently the condition of complete plasticity, may be employed only when $\epsilon_3 \neq$ constant. Therefore the condition of complete plasticity should be expected to yield good results in cases when the state of stress differs substantially from the state of plane strain.

II. Consider the solution analogous to that of Hill, presented in II of Section 1 above.

Let

$$\tau_{\rho z} = \rho, \qquad u = -\rho$$

Then

$$\sigma_\rho - \sigma_z = 2\mu_1 \sqrt{1-\rho^2}, \qquad \sigma_\rho - \sigma_\theta = \mu_1 \sqrt{1-\rho^2} - 1$$

It is easy to find

$$\sigma_\rho = -2z - \mu_1 \left[\sqrt{1-\rho^2} - \ln \frac{1+\sqrt{1-\rho^2}}{\rho}\right] + \ln \rho + C_1$$

$$\sigma_z = -2z + \mu_1 \left[3\sqrt{1-\rho^2} - \ln \frac{1+\sqrt{1-\rho^2}}{\rho}\right] + \ln \rho + C_1$$

$$w = -2z - \mu_1 \left[\sqrt{1-\rho^2} - \ln \frac{1+\sqrt{1-\rho^2}}{\rho}\right] + \ln \rho + C_2$$

Characteristic equations will be written in the form

$$z = -\sqrt{1-\rho^2} + 2\tan^{-1}\sqrt{\frac{1-\rho}{1+\rho}} + \text{const}$$

$$z = -\sqrt{1-\rho^2} - 2\tan^{-1}\sqrt{\frac{1+\rho}{1-\rho}} + \text{const}$$

III. Assume that

$$\tau_{\rho z} = \frac{1}{\rho}, \quad u = -\frac{1}{\rho}$$

It is easy to determine that in such a case the components of stress satisfying equations (1.1) and (2.1) will have the form

$$\sigma_\rho = -\mu_1\left[\sqrt{1-\frac{1}{\rho^2}} - \ln(\rho + \sqrt{\rho^2-1})\right] + \ln\rho + C_1$$
$$\sigma_z = \mu_1\left[\sqrt{1-\frac{1}{\rho^2}} + \ln(\rho + \sqrt{\rho^2-1})\right] + \ln\rho + C_1$$

Further, one easily obtains

$$w = -\mu_2\left[\sqrt{1-\frac{1}{\rho^2}} - \ln(\rho + \sqrt{\rho^2-1})\right] + C_2$$

The characteristic equations are

$$z = \pm\sqrt{\rho^2-1} + \ln(\rho + \sqrt{\rho^2-1}) + \text{const}$$

IV. One may obtain a solution which would generalize solutions developed in II and III of Section 2 above. Let

$$\tau_{\rho z} = m_1\rho + \frac{m_2}{\rho}, \qquad u = n_1\rho + \frac{n_2}{\rho} \tag{2.6}$$

then from plasticity condition (2.1) we obtain

$$\sigma_\rho - \sigma_\theta = \mu_1\sqrt{1-\left(m_1\rho + \frac{m_2}{\rho}\right)^2 - 1} \tag{2.7}$$

Substituting expressions (2.6) and (2.7) into equations (1.1), we obtain

$$\sigma_\rho = -2m_1 z - \mu_1\int\sqrt{\rho^2-(m_1\rho^2+m_2)^2}\,\frac{d\rho}{\rho^2} + C_1$$
$$\sigma_z = -2m_1 z - \mu_1\int\sqrt{\rho^2-(m_1\rho^2+m_2)^2}\,\frac{d\rho}{\rho^2} - \frac{2\mu_1}{\rho}\sqrt{\rho^2-(m_1\rho^2+m_2)^2} + C_1$$

It is also easy to find that

$$w = -2n_1 z + \int\frac{\partial u}{\partial\rho}\,\frac{\tau_{\rho z}}{(\sigma_\rho-\sigma_z)}\,d\rho + C_2$$

The characteristic equations may be found in an analogous manner.

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